

THE SPACE OF SHORT ROPES AND THE CLASSIFYING SPACE OF THE SPACE OF LONG KNOTS

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ABSTRACT. We prove affirmatively the conjecture raised by J. Mostovoy [3]; the space of *short ropes* is weakly equivalent to the classifying space of the topological monoid (or category) of long knots in \mathbb{R}^3 . The proof makes use of techniques developed by S. Galatius and O. Randal-Williams [2].

1. INTRODUCTION

A *long j -embedding* in \mathbb{R}^n is an embedding $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ that coincides with the standard inclusion outside a compact set. The space $\text{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ of all long j -embeddings in \mathbb{R}^n equipped with the C^∞ -topology is widely studied in recent years, in particular in the (meta-)stable range of dimensions. Perhaps one of the most fascinating case is $(n, j) = (3, 1)$, but this case is not in the stable range and some methods for studying $\text{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ in high (co)dimensional cases can apply to $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ to get only the information on $K := \pi_0(\text{Emb}(\mathbb{R}^1, \mathbb{R}^3))$. The group completion $\Omega B\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ would be strictly better from homotopy-theoretic view than $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ itself, because K is just a free commutative monoid with respect to the connected-sum. In fact the group completion should be a 2-fold loop space, since the little 2-disks operad acts on $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ [1].

From this viewpoint the result of [3] is very curious though it also concerned with K ; the fundamental group of the space of “short ropes” is isomorphic to $\pi_1 BK$, the group completion of K . This leads us to the question [3, Conjecture 1]; is the space of “short ropes” the classifying space $B\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ of $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$?

Our main result asserts that this is the case. In [3] a “short rope” is an embedding $r : [0, 1] \hookrightarrow \mathbb{R}^3$ of length < 3 with $r(i) = (i, 0, 0)$ for $i = 0, 1$. For such an embedding there exists at least one $t \in (0, 1)$ such that $r([0, 1])$ intersects $\{t\} \times \mathbb{R}^2$ transversely at exactly one point. In this paper we define *short ropes* as 1-manifolds satisfying the latter condition, and we show that the space of short ropes is weakly equivalent to the classifying space of the category \mathcal{K} , whose morphism space is equivalent to $\text{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. Lastly we observe that the space of “short ropes” in [3] is homotopy equivalent to the space of our short ropes.

The proof is similar to those in [2] that determine the homotopy types of the classifying spaces of various categories of cobordisms. In §3 we introduce the category \mathcal{K} and characterize the weak homotopy type of $B\mathcal{K}$ as that of certain space of 1-dimensional submanifolds. To do this we introduce some auxiliary posets of 1-manifolds with some “cylindrical parts”. Similar constructions for the space of short ropes are done in §4 and the proof is completed by comparing the auxiliary posets for long knots and short ropes.

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2. PRELIMINARIES

2.1. Notations. Throughout this paper D^m and \overline{D}^m stand respectively for the open and closed unit m -disks;

$$D^m := \{p \in \mathbb{R}^m \mid |p| < 1\}, \quad \overline{D}^m := \{p \in \mathbb{R}^m \mid |p| \leq 1\}.$$

For a 1-dimensional manifold $M \subset \mathbb{R}^1 \times D^2$ and a subset $A \subset \mathbb{R}^1$, let

$$M|_A := M \cap (A \times D^2).$$

For a one point set $A = \{T\}$, we simply write $M|_T$ for $M|_{\{T\}}$, and we regard $M|_T$ as a point in D^2 in an obvious way.

Definition 2.1. A 1-dimensional manifold $M \subset \mathbb{R}^1 \times D^2$ is said to be

- *transverse at $T \in \mathbb{R}^1$* if M transversely intersects $\{T\} \times D^2$ at a *one point set* $M|_T$,
- *cylindrical at $T \in \mathbb{R}^1$* if $M|_T$ is a one point set and there exists an $\epsilon > 0$ satisfying

$$M|_{(T-\epsilon, T+\epsilon)} = (T - \epsilon, T + \epsilon) \times M|_T.$$

2.2. Classifying spaces of categories. For a topological category C , its *nerve* is the simplicial space whose level l space $N_l C$ consists of sequences of composable l morphisms $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_l} x_l)$ in C and is topologized as a subspace of the l -th power of the total space of all morphisms in C . By definition $N_0 C$ is the space of objects in C . The face maps are given by the compositions, and the degeneracy maps are given by inserting the identity morphisms. The *classifying space* BC of C is defined as the geometric realization of $N_* C$;

$$BC := |N_* C| := \left(\bigsqcup_{l \geq 0} (N_l C \times \Delta^l) \right) / \sim,$$

where $\Delta^l := \{(\lambda_0, \dots, \lambda_l) \in [0, 1]^l \mid \sum_i \lambda_i = 1\}$ is the standard l -simplex. The relation \sim is defined so that, for any order preserving map $\sigma : \{0, \dots, l \pm 1\} \rightarrow \{0, \dots, l\}$,

$$(2.1) \quad N_{l \pm 1} C \ni (\sigma^* f, \lambda) \sim (f, \sigma_* \lambda) \in N_l C$$

where σ_* and σ^* are the induced maps on (co)simplicial spaces.

Recall from [4] a sufficient condition for a simplicial map to induce a homotopy equivalence on geometric realizations.

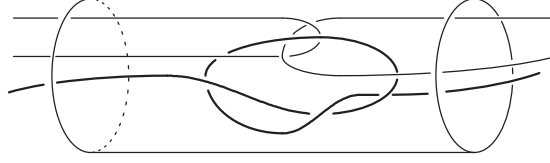
Definition 2.2 ([4, Definition A.4]). We say a simplicial space A_* is *good* if $s_i A_l \hookrightarrow A_{l+1}$ is a closed cofibration for each l and $0 \leq i \leq l$, where s_i stands for the i -th degeneracy map.

Lemma 2.3 ([4, Proposition A.1]). *Let A_* and B_* be good simplicial spaces. Suppose there exists a simplicial map $f_* : A_* \rightarrow B_*$ which is a levelwise homotopy equivalence, that is $f_l : A_l \rightarrow B_l$ is a homotopy equivalence for each l . Then f induces a homotopy equivalence $|f_*| : |A_*| \xrightarrow{\sim} |B_*|$ on the geometric realizations.*

3. THE SPACE OF LONG KNOTS AS A TOPOLOGICAL CATEGORY

Definition 3.1. Let ψ be the set of 1-dimensional submanifolds $M \subset \mathbb{R} \times D^2$ satisfying

- M is a closed subspace in $\mathbb{R}^1 \times D^2$ and $\partial M = \emptyset$,
- there exists exactly one connected component M_0 satisfying $M_0|_t \neq \emptyset$ for any $t \in \mathbb{R}^1$ (such a component is said to be *long*),
- the other connected components are (if exist) *long in either left or right*; we say a component M_1 is *long in the left* (resp. *right*) if there exists $T \in \mathbb{R}$ such that $M_1|_s \neq \emptyset$ for any $s \leq T$ (resp. $s \geq T$) but $M_1|_{(T, \infty)} = \emptyset$ (resp. $M_1|_{(-\infty, T)} = \emptyset$)

FIGURE 3.1. An element of ψ ; the long component is drawn with a thick curve

(see Figure 3.1). We call a non-long component *one-side long*. The set ψ is topologized as a subspace of $\Psi_1(\mathbb{R}^1 \times D^2)$ from [2, §2.1].

Remark 3.2. Roughly speaking, two manifolds $M, N \in \psi$ are “close to each other if they are close in a compact set”. A bit more precisely, for $M \in \psi$, the set of manifolds whose intersections with some compact set are obtained by shifting M along normal sections to M close to zero, is a basic open neighborhood of M in ψ . It would be worth mentioning that, for example, a family $M(t) \in \psi$ ($0 \leq t < 1$) satisfying $M(t)|_{[-t/(1-t), t/(1-t)]} = [-t/(1-t), t/(1-t)] \times \{0\}$ converges to the trivial long knot $\mathbb{R}^1 \times \{0\}$ in this topology as t tends to 1 (see also [2, Example 2.2]).

Definition 3.3. We define the category \mathcal{K} of *long knots* as follows. The set of objects of \mathcal{K} is D^2 with the usual topology. A non-identity morphism from p to q is a pair (T, M) , where $T > 0$ and $M \in \psi$ is a long knot from p to q , namely a connected 1-manifold (and hence is long) that is cylindrical at any $t \in (-\infty, \epsilon) \cup (T - \epsilon, \infty)$ for some $\epsilon > 0$;

$$M|_{(-\infty, \epsilon)} = (-\infty, \epsilon) \times \{p\}, \quad M|_{(T-\epsilon, \infty)} = (T-\epsilon, \infty) \times \{q\}.$$

The identity morphism $\text{id} : p \rightarrow p$ is given by $(0, \mathbb{R}^1 \times \{p\})$. The total space $\bigcup_{p,q} \text{Map}_{\mathcal{K}}(p, q)$ of all morphisms is topologized as a subspace of $(\{0\} \sqcup \mathbb{R}_{>0}) \times \psi$, where $\{0\} \sqcup \mathbb{R}_{>0}$ is a disjoint union. The composition $\circ : \text{Map}_{\mathcal{K}}(q, r) \times \text{Map}_{\mathcal{K}}(p, q) \rightarrow \text{Map}_{\mathcal{K}}(p, r)$ is defined by

$$(T_1, M_1) \circ (T_0, M_0) := (T_0 + T_1, M_0|_{(-\infty, T_0]} \cup (M_1|_{[0, \infty)} + T_0 \mathbf{e}_1)),$$

where $\mathbf{e}_1 = (1, 0, 0) \in \mathbb{R}^3$ and $+T\mathbf{e}_1$ stands for the translation by T in the direction of \mathbb{R}^1 .

In this section we show that $B\mathcal{K}$ is weakly equivalent to the subspace $\psi_s \subset \psi$ defined below. The following posets play roles as interfaces between them.

Definition 3.4. Define a poset \mathcal{D} by

$$\mathcal{D} := \{(T, M) \in \mathbb{R} \times \psi \mid M \text{ is transverse at } T\}$$

(see Definition 2.1) and topologize \mathcal{D} as a subspace of $\mathbb{R} \times \psi$. Define the partial order \leq on \mathcal{D} so that $(T, M) < (T', M')$ if and only if $M = M'$ and $T < T'$. We regard \mathcal{D} as a small category in the usual way, namely $\text{Map}_{\mathcal{D}}(x, y)$ is a one point set if $x \leq y$, and \emptyset otherwise. The total space of all morphisms is topologized as a subspace of $(\Delta \sqcup (\mathbb{R} \times \mathbb{R} \setminus \Delta)) \times \psi$, where $\Delta := \{(x, x) \in \mathbb{R} \times \mathbb{R}\}$ is the diagonal set.

Define \mathcal{D}^\perp as a subposet of \mathcal{D} consisting of (T, M) with M being cylindrical at T .

Remark 3.5. For $(T, M) \in \mathcal{D}$, the one-side long components of M are “separated” from each other; namely the components of M that are long in the left (resp. right) are contained in $(-\infty, T) \times D^2$ (resp. $(T, \infty) \times D^2$).

Notice that any element of $N_l \mathcal{D}$ ($l \geq 0$) can be expressed as a pair $(T_0 \leq \dots \leq T_l; M)$, where $M \in \psi$ is transverse at T_i for each i . This is an element of $N_l \mathcal{D}^\perp$ if moreover M is cylindrical at each T_i . Similarly any element of $N_l \mathcal{K}$ ($l \geq 1$) is of the form $(0 \leq T_1 \leq \dots \leq T_l; M)$, where M is a long knot that is cylindrical at each T_i .

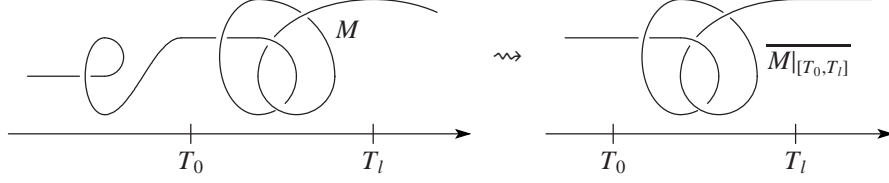


FIGURE 3.2. Cut-off and long-extension

Lemma 3.6. *The simplicial spaces $N_*\mathcal{K}$, $N_*\mathcal{D}$ and $N_*\mathcal{D}^\perp$ are good.*

Proof. For $0 \leq i \leq l$, $s_i N_i \mathcal{K} = \{(0 \leq T_1 \leq \dots \leq T_{l+1}; M) \mid T_i = T_{i+1}\} \subset N_{l+1} \mathcal{K}$ (here $T_0 := 0$) is a union of connected components of sequences involving identity morphisms, and hence the pair $(N_{l+1} \mathcal{K}, s_i N_i \mathcal{K})$ has the homotopy extension property. The proofs for $N_*\mathcal{D}$ and $N_*\mathcal{D}^\perp$ are the same. \square

Proposition 3.7. *There exists a zig-zag of levelwise homotopy equivalences $N_*\mathcal{K} \leftarrow N_*\mathcal{D}^\perp \rightarrow N_*\mathcal{D}$. Consequently $B\mathcal{K} \leftarrow B\mathcal{D}^\perp \rightarrow B\mathcal{D}$ are all homotopy equivalences.*

Proof. The proof is the same as in [2, Theorem 3.9]. That $B\mathcal{D}^\perp \rightarrow B\mathcal{D}$ induced by the inclusion is a homotopy equivalence follows from [2, Lemma 3.4], which states that, for any $(T_0 \leq \dots \leq T_l; M) \in N_l \mathcal{D}$, M can be made cylindrical at T_i in a canonical way.

Define the functor $F : \mathcal{D}^\perp \rightarrow \mathcal{K}$ on objects by $(T, M) \mapsto M|_T$, and on morphisms by

$$F(T_0 \leq \dots \leq T_l; M) := (0 \leq T_1 - T_0 \leq \dots \leq T_l - T_0; \overline{M|_{[T_0, T_l]}} - T_0 e_1),$$

where $\overline{M|_{[T_0, T_l]}}$ is the long-extension of $M|_{[T_0, T_l]}$ (see Figure 3.2), namely

$$(3.1) \quad \overline{M|_{[T_0, T_l]}} := ((-\infty, T_0] \times M|_{T_0}) \cup M|_{[T_0, T_l]} \cup ([T_l, \infty) \times M|_{T_l})$$

(this is the same as $(\varphi_\infty(T_0, T_l) \times \text{id})^{-1}(M)$ in [2, §3.2]). Notice that $M|_{[T_0, T_l]}$ is a connected subspace of the long component of M , and its long extension is also connected. This induces a map $F : N_*\mathcal{D}^\perp \rightarrow N_*\mathcal{K}$ of simplicial spaces.

We have a map $G : N_*\mathcal{K} \rightarrow N_*\mathcal{D}^\perp$, defined in level 0 by $G(p) := (0, \mathbb{R}^1 \times \{p\})$, and by the natural inclusion in positive levels (letting $T_0 := 0$). This is just a map of simplicial spaces up to homotopy (in levels 0 and 1), but is a levelwise homotopy inverse to F ; the composite $F \circ G$ is the identity, and the other composite $G \circ F$ is given by

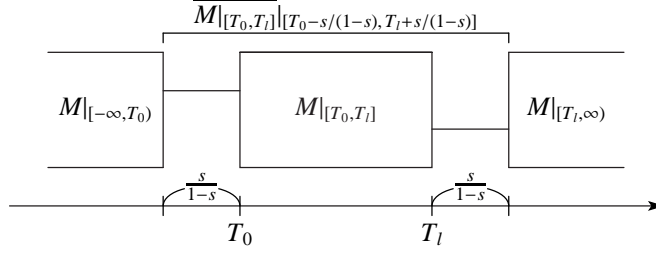
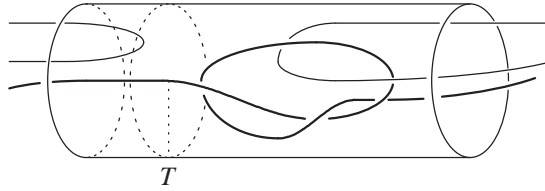
$$(3.2) \quad G \circ F(T_0 \leq \dots \leq T_l; M) = (0 \leq T_1 - T_0 \leq \dots \leq T_l - T_0; \overline{M_0|_{[T_0, T_l]}})$$

which can be homotoped to the identity by the homotopy

$$h_s(T_0 \leq \dots \leq T_l; M) := \left((1-s)T_0 \leq T_1 - sT_0 \leq \dots \leq T_l - sT_0; \left(M|_{(-\infty, T_0]} - \frac{s}{1-s} e_1 \right) \cup \overline{M|_{[T_0, T_l]}}|_{[T_0-s/(1-s), T_l+s/(1-s)]} \cup \left(M|_{[T_l, \infty)} + \frac{s}{1-s} e_1 \right) - sT_0 e_1 \right)$$

(see Figure 3.3). This homotopy h_s extends the cylindrical parts $M|_{(T_0-\epsilon, T_0]}$ and $M|_{[T_l, T_l+\epsilon)}$ respectively to left and right so that $M|_{(-\infty, T_0]}$ and $M|_{[T_l, \infty)}$ (in which all the one-side long components are contained) escape respectively to “ $\{\mp\infty\} \times D^2$,” and translates whole manifold by $-T_0$ in the direction of \mathbb{R}^1 . By definition $h_0 = \text{id}$, and h_1 equals (3.2); $M|_{(-\infty, T_0]}$ and $M|_{[T_l, \infty)}$ “vanish” at $s = 1$ by definition of the topology of ψ (see Remark 3.2).

Therefore $F : N_*\mathcal{D}^\perp \rightarrow N_*\mathcal{K}$ is a levelwise homotopy equivalence of good simplicial spaces (Lemma 3.6), and $B\mathcal{D}^\perp \rightarrow B\mathcal{K}$ is a homotopy equivalence by Lemma 2.3. \square


 FIGURE 3.3. The homotopy h_s in the proof of Proposition 3.7

 FIGURE 3.4. An element of ψ_s ; the long component is drawn with a thick curve

Definition 3.8. Define the subspace $\psi_s \subset \psi$ as consisting of $M \in \psi$ such that there exists $T \in \mathbb{R}$ with M transverse at T (see Figure 3.4).

In particular one-side long components of $M \in \psi_s$ are “separated”, namely there exists $T \in \mathbb{R}$ such that all the components of M that are long in the left (resp. right) are in $(-\infty, T) \times D^2$ (resp. $(T, \infty) \times D^2$).

Following [2], we denote an element $((T_0 \leq \dots \leq T_l; M), (\lambda_0, \dots, \lambda_l)) \in N_l \mathcal{D}$ by a formal sum $\sum_{0 \leq i \leq l} \lambda_i T_i$ (this notation is compatible with the structure maps of nerves of posets).

Theorem 3.9. The forgetful map $N_l \mathcal{D} \rightarrow \psi_s$, $\sum_i \lambda_i T_i \mapsto M$, induces a weak homotopy equivalence $u : B\mathcal{D} \xrightarrow{\sim} \psi_s$. Thus $B\mathcal{K}$ is weakly equivalent to ψ_s .

Proof. The proof is the same as that of [2, Theorem 3.10]: Given the following commutative diagram of strict arrows,

$$\begin{array}{ccc} \partial \overline{D}^m & \xrightarrow{f} & B\mathcal{D} \\ \downarrow & \searrow g & \downarrow u \\ \overline{D}^m & \xrightarrow{f} & \psi_s \end{array}$$

we find a dotted $g : \overline{D}^m \rightarrow B\mathcal{D}$ that makes the diagram commutative. This means that the relative homotopy group $\pi_m(\psi'_s, B\mathcal{D})$ (ψ'_s is the mapping cylinder of u) vanishes for all m , and u induces an isomorphism of homotopy groups in any dimension.

For $a \in \mathbb{R}$, let $U_a \subset \overline{D}^m$ be the set of $x \in \overline{D}^m$ such that $f(x) \in \psi$ is transverse at a . This is an open subset of \overline{D}^m and $\{U_a\}_{a \in \mathbb{R}}$ is an open covering of \overline{D}^m because, by definition, such an a exists for any $M \in \psi_s$. So by compactness we can pick finitely many $a_0 < \dots < a_k$ such that $\{U_{a_i}\}_{1 \leq i \leq k}$ covers \overline{D}^m . Pick a partition of unity $\{\lambda_i : \overline{D}^m \rightarrow [0, 1]\}_{1 \leq i \leq k}$ subordinate to the cover. Using λ_i as a formal coefficient of a_i gives a map

$$\hat{g} : \overline{D}^m \rightarrow B\mathcal{D}, \quad \hat{g}(x) := \sum_{0 \leq i \leq k} \lambda_i(x) a_i$$

(represented by elements in $N_k \mathcal{D} \times \Delta^k$) which lifts f , namely $u \circ \hat{g} = f$. Now we produce a homotopy $h : [0, 1] \times \partial \overline{D}^m \rightarrow B\mathcal{D}$ such that $h(0, -) = \hat{g}|_{\partial \overline{D}^m}(-)$, $h(1, -) = \hat{f}(-)$ and $h(s, -)$ lifts $f|_{\partial \overline{D}^m}$ for all s ; if such an h exists, then we can define the desired map g by

$$g(x) := \begin{cases} \hat{g}(2x) & |x| \leq 1/2, \\ h(2|x| - 1, x/|x|) & |x| \geq 1/2. \end{cases}$$

Since \hat{f} is also a lift of $f|_{\partial \overline{D}^m}$, we may suppose that \hat{f} is of the form

$$\hat{f}(x) = \sum_{0 \leq i \leq l} \mu_i(x) b_i$$

for some $\mu_0, \dots, \mu_l \geq 0$, $\sum_i \mu_i(x) = 1$ and $b_0 < \dots < b_l$ (underlying manifold M for \hat{f} is the same as that for f). Let $c_0 < \dots < c_n$ be the re-ordering of the set $\{a_i\}_i \cup \{b_j\}_j$ in ascending order. Using the relation (2.1) we can write $\hat{g}|_{\partial \overline{D}^m}$ and \hat{f} as

$$\begin{aligned} \hat{g}|_{\partial \overline{D}^m}(x) &= \sum_{0 \leq i \leq n} \alpha_i(x) c_i \quad \text{for some } \alpha_0, \dots, \alpha_n \geq 0, \quad \sum_i \alpha_i = 1, \\ \hat{f}(x) &= \sum_{0 \leq i \leq n} \beta_i(x) c_i \quad \text{for some } \beta_0, \dots, \beta_n \geq 0, \quad \sum_i \beta_i = 1 \end{aligned}$$

(represented by elements in $N_n \mathcal{D} \times \Delta^n$). We define h using the affine structure on the fibers of u ;

$$h(s, x) := s\hat{g}|_{\partial \overline{D}^m}(x) + (1-s)\hat{f}(x) := \sum_{0 \leq i \leq n} (s\alpha_i(x) + (1-s)\beta_i(x))c_i. \quad \square$$

Remark 3.10. We have topologized the spaces of morphisms of various categories so that the identity morphisms form disjoint components, as was also done in [2]. We may instead topologize the total space of morphisms in \mathcal{K} (resp. \mathcal{D}) as a subspace of $[0, \infty) \times \psi$ (resp. $\mathbb{R} \times \mathbb{R} \times \psi$) and with the latter topology we can prove the similar results to the above. An advantage of the former topology is that the proof of goodness of the nerves becomes easier.

4. THE SPACE OF SHORT ROPES

In this section we characterize the weak homotopy type of $B\mathcal{K}$ as that of the space of *short ropes*.

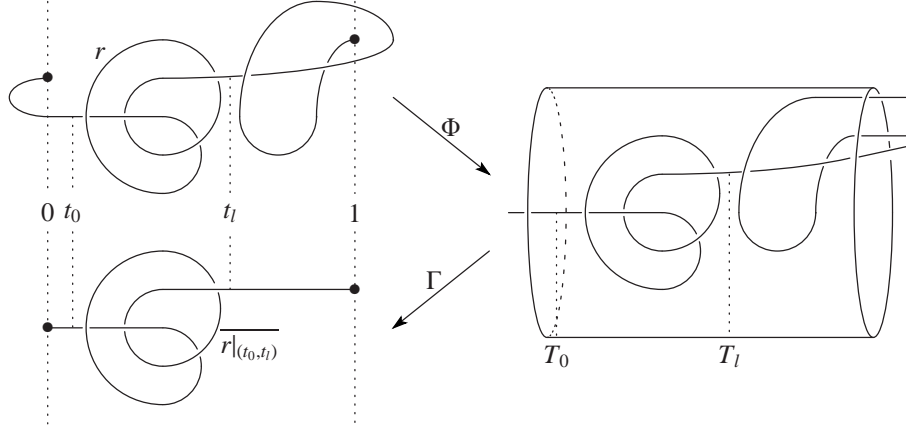
Definition 4.1 ([3]). A *rope* is a compact, connected 1-dimensional submanifold $r \subset \mathbb{R}^1 \times D^2$ with non-empty boundary $\partial r = \{r_0, r_1\}$, $r_i \in \{i\} \times D^2$. A rope r is said to be *short* if there exists $t \in (0, 1)$ such that r is transverse at t . Let R be the set of all short ropes, topologized similarly to ψ .

The function $f(t) := \tan \pi(t - (1/2))$ gives an orientation preserving diffeomorphism $f : (0, 1) \xrightarrow{\cong} \mathbb{R}$. Define the “cut-off” map $c : R \rightarrow \psi_s$ by

$$c(r) := (f \times \text{id}_{D^2})(r \cap ((0, 1) \times D^2)).$$

This map is defined since, for any short rope r , exactly one “long” component should be contained in $r \cap ((0, 1) \times D^2)$.

Our aim is to show that c is a weak equivalence, and for this we introduce the following posets as interfaces between R and ψ_s .

FIGURE 4.1. The maps Φ and Γ

Definition 4.2. Define a poset \mathcal{E} by

$$\mathcal{E} := \{(t, r) \in (0, 1) \times R \mid r \text{ is transverse at } t\}.$$

Define the partial order \leq on \mathcal{E} so that $(t, r) < (t', r')$ if and only if $r = r'$ and $t < t'$. We regard \mathcal{E} as a small category in the same way as \mathcal{D} . The total space of all morphisms is topologized as a subspace of $(\Delta \sqcup ((0, 1) \times (0, 1) \setminus \Delta)) \times R$.

Define \mathcal{E}^\perp as a subposet of \mathcal{E} consisting of (t, r) with r being cylindrical at t .

Lemma 4.3. *The simplicial spaces $N_*\mathcal{E}$ and $N_*\mathcal{E}^\perp$ are good.*

Proof. The same as that of Lemma 3.6. \square

Any element in $N_l\mathcal{E}$ can be expressed as a pair $(t_0 \leq \dots \leq t_l; r)$ where $0 < t_i < 1$ and $r \in R$ is transverse at t_i for each i .

Proposition 4.4. *There exists a zig-zag of levelwise homotopy equivalences $N_*\mathcal{E} \leftarrow N_*\mathcal{E}^\perp \rightarrow N_*\mathcal{D}^\perp$. Consequently $B\mathcal{E}$ is weakly homotopy equivalent to $B\mathcal{D}$.*

Proof. That the inclusion $\mathcal{E}^\perp \rightarrow \mathcal{E}$ induces a homotopy equivalence $B\mathcal{E}^\perp \xrightarrow{\cong} B\mathcal{E}$ follows in the same way as [2, Theorem 3.9], using [2, Lemma 3.4].

Let $f(t) := \tan \pi(t - (1/2))$. For $l \geq 0$, define $\Phi : N_l\mathcal{E}^\perp \rightarrow N_l\mathcal{D}^\perp$ by

$$\Phi(t_0 \leq \dots \leq t_l; r) := (f(t_0) \leq \dots \leq f(t_l); c(r))$$

(see Figure 4.1). Define the map in the reverse direction $\Gamma : N_l\mathcal{D}^\perp \rightarrow N_l\mathcal{E}^\perp$ by

$$\Gamma(T_0 \leq \dots \leq T_l; M) := (t_0 \leq \dots \leq t_l; (f^{-1} \times \text{id}_{D^2})(\overline{M|_{[T_0, T_l]}})),$$

where $\overline{M|_{[T_0, T_l]}}$ is the long-extension of $M|_{[T_0, T_l]}$ (see (3.1)), and $t_i := f^{-1}(T_i) \in (0, 1)$ (see Figure 4.1). Notice that $r := (f^{-1} \times \text{id}_{\mathbb{R}^2})(\overline{M|_{[T_0, T_l]}})$ is a tame submanifold in $(0, 1) \times D^2$ since $\overline{M|_{[T_0, T_l]}}$ is a compact manifold attached by straight half-lines $(-\infty, T_0] \times M|_{T_0}$ and $[T_l, \infty) \times M|_{T_l}$ that are mapped by $f^{-1} \times \text{id}_{D^2}$ respectively to segments $(0, t_0] \times r|_{t_0}$ and $[t_l, 1) \times r|_{t_l}$.

Clearly Φ is a simplicial maps. We show that Φ is a levelwise homotopy equivalence, with a homotopy inverse Γ . The composite $\Phi \circ \Gamma$ is given by

$$\Phi \circ \Gamma(T_0 \leq \dots \leq T_l; M) = (T_0 \leq \dots \leq T_l; \overline{M|_{[T_0, T_l]}})$$

and a similar isotopy to h_s from the proof of Proposition 3.7 proves that $F \circ G \simeq \text{id}$.

The other composite $\Gamma \circ \Phi$ is given by

$$\Gamma \circ \Phi(t_0 \leq \cdots \leq t_l; r) := (t_0 \leq \cdots \leq t_l; \overline{r|_{(t_0, t_l)}}),$$

where

$$\overline{r|_{(t_0, t_l)}} := ([0, t_0] \times r|_{t_0}) \cup r|_{(t_0, t_l)} \cup ([t_l, 1] \times r|_{t_l}) \in R$$

is the “long-extension” of $r|_{(t_0, t_l)}$. The rope $\overline{r|_{(t_0, t_l)}}$ can be obtained from r by unknotting the edge parts $r|_{(-\infty, t_0)} \sqcup r|_{(t_l, \infty)}$. This can be done in a canonical way, as explained in the proof of [3, Lemma 10]. In fact, $r|_{(-\infty, t_0)}$ is (a variant of) a “rope which extends to a (long) knot without singularities to the right” [3, p. 440] and the space W_R of such ropes is contractible [3, Lemma 10]. Similarly $r|_{(t_l, \infty)}$ can be unknotted in a canonical way. Moreover the contracting isotopy given in [3] can be taken so that it transforms ropes inside $((-\infty, t_0) \sqcup (t_l, \infty)) \times D^2$ and $r|_{[t_0, t_l]}$ remains unchanged. Thus $\overline{r|_{(t_0, t_l)}}$ can be transformed to r remaining to be cylindrical at t_i , and hence $\Gamma \circ \Phi$ is homotopic to the identity. \square

Theorem 4.5. *The forgetful map induces a weak equivalence $v : B\mathcal{E} \rightarrow R$.*

Proof. Replace \mathcal{D} with \mathcal{E} and take a from $(0, 1)$ in the proof of Theorem 3.9. \square

Corollary 4.6. *There exist a commutative diagram consisting of weak equivalences*

$$\begin{array}{ccccc} R & \xrightarrow{\sim} & \psi_s & & \\ \uparrow v' \sim & & \uparrow u' \sim & & \\ B\mathcal{E}^\perp & \xrightarrow[\sim]{\Phi} & B\mathcal{D}^\perp & \xrightarrow[\sim]{F} & B\mathcal{K} \end{array}$$

where u', v' are the composites of u, v with the inclusions.

As a final remark we outline a construction of a weak equivalence between the space R of our short ropes and that of “short ropes” in the sense of Mostovoy [3].

A “short rope” in [3] is an embedding $r : [0, 1] \hookrightarrow \mathbb{R}^3$ of length < 3 satisfying $r(0) = (0, 0, 0)$, $r(1) = (1, 0, 0)$. Let B_2 be the space of such embeddings. Notice that the image of such an embedding satisfies the condition of Definition 4.1. Thus if we define B' as the space of embeddings whose images are short ropes in the sense of Definition 4.1, then we have an inclusion $B_2 \hookrightarrow B'$.

There is a continuous map $B' \rightarrow R$ defined by $r \mapsto r([0, 1])$ and it can be seen that this map is a homotopy equivalence; a homotopy inverse $R \rightarrow B'$ is given by a certain diffeomorphism $\text{id}_{\mathbb{R}^1} \times \xi : \mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^1 \times D^2$ ($\xi : \mathbb{R}^2 \xrightarrow{\cong} D^2$ depends on r) mapping $r \cap (\{i\} \times D^2)$ to $(i, 0, 0)$, followed by parametrizing the resulting rope so that it is of constant velocity. The same argument shows that there exists a homotopy equivalence $B_2 \rightarrow \bar{R}$, where \bar{R} is the space of ropes in the sense of Definition 4.1 of length < 3 ; the diffeomorphism $\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^1 \times D^2$ can be chosen so that it does not increase the length of ropes.

Next let $\bar{\mathcal{E}}$ be a subposet of \mathcal{E} consisting of (t, r) with r of length < 3 . Then we have a commutative diagram

$$\begin{array}{ccccc} \bar{B\mathcal{E}} & \xrightarrow{\bar{v}} & \bar{R} & \xleftarrow{\cong} & B_2 \\ \downarrow & & \downarrow \cap & & \downarrow \cap \\ B\mathcal{E} & \xrightarrow[\text{Theorem 4.5}]{\cong} & R & \xleftarrow{\cong} & B' \end{array}$$

where $B\bar{\mathcal{E}} \rightarrow B\mathcal{E}$ and \bar{v} are induced respectively from the inclusion and the forgetful map. We claim that $B\bar{\mathcal{E}} \rightarrow B\mathcal{E}$ and \bar{v} are both (weak) homotopy equivalence; the proof for \bar{v} is the same as Theorem 4.5, and $\bar{\mathcal{E}} \rightarrow \mathcal{E}$ induces a levelwise homotopy equivalence $N_*\bar{\mathcal{E}} \rightarrow N_*\mathcal{E}$. Indeed a retraction $N_l\mathcal{E} \rightarrow N_l\bar{\mathcal{E}}$ is given by firstly unknotting $r|_{(-\infty, t_0]} \sqcup r|_{[t_l, \infty)}$ similarly to the proof of Proposition 4.4, then shrinking the resulting rope $r|_{(t_0, t_l)}$ in a suitable way so that its length becomes less than 3.

Therefore the space R of our short ropes is weakly equivalent to the space B_2 of “short ropes” in [3].

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